Lecture 23

15.2 - Iterated Integrals

Recall that, over a rectangle $R=[a,b]\times[c,d]$, the double integral of $f(x,y)$ is:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

We can do this two ways:

- integrate over x first:

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\lim_{m\to\infty} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta x \right) \Delta y$$

(There are some technicalities here, but we'll ignore them.)

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left(\int_a^b f(x, y_j^*)\,dx \right) \Delta y$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

(which we usually write as)

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- integrate over y first:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

These are called <u>iterated integrals</u>.

To compute iterated integrals, we work <u>from the inside out</u>.

So, question one is: "how do we compute $\int_a^b f(x,y)\,dx$ or $\int_c^d f(x,y)\,dy$?" The answer is: "just as with partial derivatives, treat the other variable as a constant." This is called <u>partial integration</u>.

<u>Ex</u>: Compute $\int_0^5 12x^2y^3\,dx$ and $\int_0^1 12x^2y^3\,dy$.

<u>Sol</u>: First, $\int 12x^2y^3\,dx = 4x^3y^3 + g(y)$ ← the "constant of integration" here is a constant with respect to x (i.e. something that when you take $\frac{\partial}{\partial x}$ of it, you get 0.)

$$\boxed{\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 500y^3 - 0 = 500y^3.}$$

$\int 12x^2y^3\,dy = 3x^2y^4 + h(x)$.

$$\boxed{\int_0^1 12x^2y^3\,dy = 3x^2y^4\Big|_0^1 = 3x^2 - 0 = 3x^2}$$

◇

<u>Comment</u>: We don't have to compute the indefinite integral each time. I did it for illustrative purposes.

Let's use this to compute some double integrals:

<u>Ex</u>: Compute the following integrals:

ⓐ $\int_0^2 \int_0^4 y^3 e^{2x}\, dy\, dx$ ⓑ $\int_0^4 \int_0^2 y^3 e^{2x}\, dx\, dy$

<u>Sol</u>:

ⓐ $\int_0^2 \int_0^4 y^3 e^{2x}\, dy\, dx = \int_0^2 \left(\frac{1}{4} y^4 e^{2x} \right)\Big|_0^4 dx$

$= \int_0^2 \left(64 e^{2x} - 0 \right) dx = \int_0^2 64 e^{2x}\, dx = 32 e^{2x} \Big|_0^2$

$= 32 e^4 - 32 e^0 = 32(e^4 - 1).$

ⓑ $\int_0^4 \int_0^2 y^3 e^{2x}\, dx\, dy = \int_0^4 \left(\frac{1}{2} y^3 e^{2x} \Big|_0^2 \right) dy$

$= \int_0^4 \left(\frac{1}{2} y^3 e^4 - \frac{1}{2} y^3 e^0 \right) dy = \int_0^4 \frac{1}{2}(e^4 - 1) y^3\, dy$

$= \frac{(e^4-1)}{8} y^4 \Big|_0^4 = \frac{(e^4-1)}{8}(256 - 0) = 32(e^4 - 1).$ ◇

The fact that these are equal is no coincidence.

<u>Fubini's Theorem</u>: If f is continuous on $R=[a,b]\times[c,d]$, then:

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the integrals exist.

<u>Ex</u>: Compute $\iint_R ye^{-xy}\,dA$ where $R=[0,2]\times[0,3]$

<u>Sol</u>:
If we integrate with respect to x first:

$$\iint_R ye^{-xy}\,dA = \int_0^3\int_0^2 ye^{-xy}\,dx\,dy = \int_0^3\left(-e^{-xy}\Big|_0^2\right)dy$$

$$= \int_0^3\left(-e^{-2y}-(-e^0)\right)dy = \int_0^3\left(1-e^{-2y}\right)dy$$

$$= \left(y+\tfrac{1}{2}e^{-2y}\right)\Big|_0^3 = \left(3+\tfrac{1}{2}e^{-6}\right)-\left(0+\tfrac{1}{2}e^0\right)$$

$$= \frac{5}{2}+\frac{1}{2}e^{-6}$$

◇

Recall, if $f(x,y) \geq 0$ on R, $\iint_R f(x,y)\,dA$ gives the volume of the solid S below $f(x,y)$ and above R.

Ex: Find the volume of the region bounded by the surface $z = 1 + e^x \sin y$, and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

Sol: Since $e^x > 0$ always, and $\sin y \geq 0$ for $0 \leq y \leq \pi$, thus $z \geq 0$ on $R = [-1,1] \times [0,\pi]$.

$$\text{Vol} = \int_{-1}^{1} \int_{0}^{\pi} (1 + e^x \sin y)\,dy\,dx = \int_{-1}^{1} \Big[y - e^x \cos y \Big]_0^{\pi} dx$$

$$= \int_{-1}^{1} \Big[\big(\pi - e^x(-1)\big) - \big(0 - e^x \cos 0\big) \Big] dx$$

$$= \int_{-1}^{1} \big[\pi + e^x + e^x \big] dx = \int_{-1}^{1} (\pi + 2e^x)\,dx = (\pi x + 2e^x)\Big|_{-1}^{1}$$

$$= (\pi + 2e) - (-\pi + 2e^{-1}) = 2\pi + 2e - 2e^{-1}$$

$\diamond$